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An Efficient Numerical Method for Solving Inverse Conduction Problem in a Hollow Cylinder

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Nomenclature

A,B,C,D	= coefficient of square temperature matrix
Bi	= Biot number, hL/k
h	= heat-transfer coefficient
k	= thermal conductivity
L	= thickness of material $(r_0 - r_i)$
$\dot{q}_c$	= surface heat flux
r	= radial coordinate
r_i	= inner radius of cylinder
r_o	= outer radius of cylinder
T	= nondimensional temperature
T^{n-1}	= nondimensional temperature at beginning of time step
T^n	= nondimensional temperature at end of time step
t	= nondimensional time, $\alpha\tau/L^2$
Δt	= computing time
X	= nondimensional radial coordinate, $(r-r_i)/L$
ΔX	= node thickness
θ	= temperature
α	= thermal diffusivity
τ	= time

Subscripts

g	= combustion gas temperature
i,o,R	= node identifier
in	= initial temperature
j	= thermocouple location
s	= surface temperature

Superscript

n	= designated point $(t+\Delta t)$

Introduction

THE calculation of surface heat flux and surface temperature from a measured temperature history at some location inside the body is called the inverse heat conduction problem. Many configurations such as spheres, slabs, and cylinders have been studied and methods such as numerical, graphical, series, convolution integral, and Laplace transforms have been utilized. An excellent discussion of previous investigations for solving the inverse problem can be found in Ref. 1. In a previous study,[2] the two-level Crank-Nicholson method was used for estimation of heat-transfer coefficient in a rocket nozzle of a finite slab thickness. But, as mentioned in Ref. 3, the equivalent slab treatment is not valid for the thickness-to-radius ratio exceeding 0.2, which is case in the vicinity of the throat region of a rocket nozzle, and always gives conservative estimates for temperatures in the cylindrical structure. This Note extends the inverse conduction problem to the cylindrical geometry using a grid point shift arrangement[4] on the boundaries. The grid point shift of surface conditions contains the following advantages as compared to implicit analog of the boundary conditions:

1) The arrangement of a grid point shifted from the boundaries is the most convenient to solve the inverse problem in cylindrical or spherical coordinates.

2) Using this arrangement, one can easily simulate mathematically and physically the most general boundary condition of the third kind.[5] It should be mentioned here that the number of unknown values of temperatures for a given number of increments is independent of the boundary conditions.

3) The surface temperature oscillation does not occur when first interior point is located one-half increment from the boundary, whereas a grid point located on the boundary requires a backward finite difference analog and also needs care in selecting the time interval in order to achieve a stable solution.

One disadvantage of the grid point shift arrangement is that the temperature at the boundaries is evaluated as the average of the exterior value and the last interior value. The exterior value can be obtained from the appropriate boundary condition analog.

This Note reports a simple numerical scheme to solve the inverse conduction problem using transient temperature data for estimating the unknown surface conditions. A general digital program is discussed that can treat a variety of boundary conditions using a single set of equations.

Analysis

Consider a long hollow cylinder with a finite wall thickness, having a heat sink at one surface and a perfect insulation at the other. The material of the cylinder is considered to be homogeneous and isotropic with constant thermophysical properties. Let r_i and r_o be, respectively, the inner and outer radii. If the temperature of the cylinder is initially uniform at θ_{in}, the mathematical problem governing the temperature may be written as

$$\frac{\partial^2 T}{\partial X^2} + \frac{1}{X+(r_i/L)}\frac{\partial T}{\partial X} = \frac{\partial T}{\partial t} \tag{1}$$

with the boundary conditions

$$\frac{\partial T(0,t)}{\partial X} = -1, \qquad t>0 \tag{2}$$

$$\frac{\partial T(1,t)}{\partial X} = 0, \qquad t>0 \tag{3}$$

and

$$T(X,0) = 0 \qquad \text{for all } X \tag{4}$$

where nondimensional temperature T is $k(\theta-\theta_{in})/\dot{q}_c L$.

The finite difference conditions of Eq. (1) with the boundary conditions of Eqs. (2) and (3) are developed by arranging the grid points as described in Ref. 4. The value of

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independent variable at each point is given by

$$X_i = (i - \tfrac{1}{2})\Delta X \tag{5}$$

In this arrangement, there are no grid points on the surfaces. However, there are two grid points outside the boundaries at each end of the surface. These are shown as X_o and X_{R+1}. Writing Eq. (1) in Crank-Nicholson two-time level implicit form at the general node i, as

$$\left[\frac{(2i-2)\Delta X + (2r_i/L)}{(2i-1)\Delta X + (2r_i/L)}\right]T_{i-1}^n + \left[-2 - \frac{2(\Delta X)^2}{\Delta t}\right]T_i^n + \left[\frac{2i\Delta X + (2r_i/L)}{(2i-1)\Delta X + (2r_i/L)}\right]T_{i+1}^n = -\left[\frac{(2i-2)\Delta X + (2r_i/L)}{(2i-1)\Delta X + (2r_i/L)}\right]T_{i-1}^{n-1} + \left[2 - \frac{2(\Delta X)^2}{\Delta t}\right]T_i^{n-1} - \left[\frac{2i\Delta X + (2r_i/L)}{(2i-1)\Delta X + (2r_i/L)}\right]T_{i+1}^{n-1} \tag{6}$$

Now we can readily obtain the governing equation of a tridiagonal system of equations of the form

$$A_i T_{i-1}^n + B_i T_i^n + C_i T_{i+1}^n = D_i, \qquad 2 \le i \le (R-1) \tag{7}$$

The finite difference analog to the boundary condition of Eq. (2) is

$$(T_1 - T_o)/\Delta X = -1 \tag{8}$$

This analog is second-order correct, since it is used to approximate the derivative at the center of the interval.

The resulting finite difference equation in the tridiagonal form at $i=1$ is

$$\left[-2 - \frac{2(\Delta X)^2}{\Delta t} + \frac{(2r_i/L)}{\Delta X + (2r_i/L)}\right]T_1^n + \left[\frac{2\Delta X + (2r_i/L)}{\Delta X + (2r_i/L)}\right]T_2^n = \left[2 - \frac{2(\Delta X)^2}{\Delta t} - \frac{(2r_i/L)}{\Delta X + (2r_i/L)}\right]T_1^{n-1} - \left[\frac{2\Delta X + (2r_i/L)}{\Delta X + (2r_i/L)}\right]T_2^{n-1} - \left[\frac{4\Delta X(2r_i/L)}{\Delta X + (2r_i/L)}\right] \tag{9}$$

An expression for T_{R+1} is derived from the boundary condition of Eq. (3) and the resulting finite difference analog at $i=R$ is given by

$$\left[\frac{(2R-2)\Delta X + (2r_i/L)}{(2R-1)\Delta X + (2r_i/L)}\right]T_{R-1}^n + \left[-2 - \frac{2(\Delta X)^2}{\Delta t} + \frac{2R\Delta X + (2r_i/L)}{(2R-1)\Delta X + (2r_i/L)}\right]T_R^n = -\left[\frac{(2R-2)\Delta X + (2r_i/L)}{(2R-1)\Delta X + (2r_i/L)}\right]T_{R-1}^{n-1} + \left[2 - \frac{2(\Delta X)^2}{\Delta t} - \frac{2R\Delta X + (2r_i/L)}{(2R-1)\Delta X + (2r_i/L)}\right]T_R^{n-1} \tag{10}$$

The tridiagonal system of Eqs. (7), (9), and (10) can be solved using the Thomas algorithm.[4] But in Eq. (9), $\dot{q}_c$ is an unknown parameter. In estimating $\dot{q}_c$, one minimizes

$$F(\dot{q}_c) = [T_c(X_j,t) - T_m(X_j,t)] \tag{11}$$

where T_c and T_m are, respectively, calculated and measured thermocouple temperatures at (X_j,t).

The Newton-Raphson method[6] is used here for estimating $\dot{q}_c$. The iteration procedure starts with an initial value of $\dot{q}_c$ and is repeated until $|F|$ is less than, say, 10^{-4}. Now, the expression for the estimation of the convective heat-transfer coefficient can be written as

$$h = \dot{q}_c / (\theta_g - \theta_s) \tag{12}$$

The surface temperature θ_s is calculated as the average of the exterior value and the last interior value. The exterior value can be obtained from the boundary condition analog as described above.

In Eq. (12), θ_g is an unknown quantity. For estimating this quantity, the governing heat conduction equation is now nondimensionalized by introducing the following variables: $T = (\theta - \theta_{in})/(\theta_g - \theta_{in})$, $X = (r - r_i)/L$, $t = \alpha\tau/L^2$, and $Bi = hL/k$; and solved with the following boundary and initial conditions:

$$\frac{\partial T(0,t)}{\partial X} = Bi[\theta(0,t) - 1], \qquad t > 0 \tag{13}$$

$$\frac{\partial T(1,t)}{\partial X} = 0, \qquad t > 0 \tag{14}$$

and

$$T(X,0) = 0 \text{ for all } X \tag{15}$$

Table 1 Comparison of present solution with the Bartz solution

τ, s	θ_s, K	T_m at outer surface, K	$\dot{q}_c \times 10^6$ W/m^2	h, W/m$^2 \cdot$K	h_B, W/m$^2 \cdot$K[a]	θ_g, K	θ_{gc}, K
6	1260.2	326	3.6805	1789.6	2254.2	3316	2946
7	1175.9	342	3.3995	1628.0	2254.2	3264	2946
8	1160.7	356	2.4745	1181.4	2254.2	3255	2946
9	1165.8	380	2.5385	1194.7	2254.2	3290	2946
10	1196.0	402	2.5348	1261.1	2254.2	3206	2946
11	1192.3	425	2.3385	1166.4	2254.2	3197	2946
12	1205.8	440	2.2094	1114.8	2254.2	3187	2946
13	1211.0	460	2.1333	1229.5	2254.2	2946	2946
14	1222.1	479	2.0441	1187.5	2254.2	2943	2946
15	1237.1	507	2.0626	1206.7	2254.2	2946	2946
16	1249.1	528	2.0027	1180.9	2254.2	2945	2946

[a] h_B = heat-transfer coefficient (Bartz).

The finite difference analog to the boundary condition [Eq. (13)] is

$$\frac{T_1 - T_o}{\Delta X} - Bi\left(\frac{T_1 + T_o}{2}\right) = -Bi \tag{16}$$

Written explicitly for T_o, it becomes

$$T_o = \left[\frac{2 - Bi(\Delta X)}{2 + Bi(\Delta X)}\right] T_1 + \frac{2Bi(\Delta X)}{2 + Bi(\Delta X)} \tag{17}$$

The resulting finite difference equation at $i = 1$ is given by

$$\left\{-2 - \frac{2(\Delta X)^2}{\Delta t} + \left[\frac{2 - Bi(\Delta X)}{2 + Bi(\Delta X)}\right]\left[\frac{(2r_i/L)}{\Delta X + (2r_i/L)}\right]\right\} T_1^n$$
$$+ \left[\frac{2\Delta X + (2r_i/L)}{\Delta X + (2r_i/L)}\right] T_2^n = \left\{2 - \frac{2(\Delta X)^2}{\Delta t} - \frac{2 - Bi(\Delta X)}{2 + Bi(\Delta X)}\right]$$
$$\times \left[\frac{(2r_i/L)}{\Delta X + (2r_i/L)}\right]\Big\} T_1^{n-1} - \left[\frac{2\Delta X + (2r_i/L)}{\Delta X + (2r_i/L)}\right] T_2^{n-1}$$
$$- \left[\frac{4Bi(\Delta X)}{2 + Bi(\Delta X)}\right]\left[\frac{(2r_i/L)}{\Delta X + (2r_i/L)}\right] \tag{18}$$

However, the finite difference equation (10) corresponding to the boundary condition [Eq. (14)] remains unchanged. The gas temperature can be obtained by using the above-mentioned minimization technique.

Example

Using the present numerical method, estimation of wall heat flux, surface temperature, convective heat-transfer coefficient, and combustion gas temperature for a typical divergent rocket nozzle made of mild steel is carried out in conjunction with experimentally measured outer surface temperature data in a static test. The nozzle condition and material properties taken are $r_i = 0.0839$ m, $r_o = 0.1050$ m, $\theta_{in} = 300$ K, $k = 35$ W/m·K (average), $\alpha = 8.1291 \times 10^{-6}$ m^2/s, and burning time = 16 s.

The combustion gas temperature θ_{gc} is obtained from thermodynamic calculations. This inverse program, in turn, utilized these transient data to determine unknown surface conditions. Twenty space intervals and a time increment of 1 s is taken for starting the solution. It is seen from Table 1 that the estimated value of the convective heat transfer is somewhat lower than the calculated results of Bartz.[7] The percentage error, $[(\theta_g - \theta_{gc})/\theta_{gc}] \times 100$ between the estimated value of θ_g and θ_{gc} is found to vary in the range of 12.1 − 0.1%. These disagreements are attributed to the higher initial time step for starting the solution. The errors may also arise due to the magnitude of the thermocouple temperatures and in relation to the surface. However, it is also important to mention here that Bartz equation gives a conservative estimate of the heat-transfer coefficient.[6,8] The comparison of the heat-transfer coefficient obtained from the present analysis is lower than the calculated results of the previous analysis.[2] It reveals that the equivalent slab treatment gives a conservative estimation of the wall thickness which, in turn, increases the structural weight of the rocket nozzle.

Conclusions

The grid point shift arrangement used in the present numerical scheme with the Newton-Raphson iteration procedure proves quite useful in estimating the values of the surface conditions from the measured thermocouple temperature measurement on the outer surface of the rocket nozzle. The advantage of using this numerical approach lies in its capability of handling different boundary conditions, geometries, and irregular behaviors of the surface heat flux. The grid point shift arrangement of the boundary condition is found efficient and numerically stable.

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